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ADDENDUM

Addendum to 'On nonlinear angular momentum theories, their representations and associated Hopf structures'

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Abstract. Further unitary irreducible representations of the Higgs algebra are constructed.

In [1], the unitary irreducible representations of the Higgs algebra [2]

$$\begin{aligned}&[J_3, J_\pm] = \pm J_\pm\\&[J_+, J_-] = 2J_3 + 8\beta J_3^3\end{aligned} \tag{1}$$

whose Casimir is given by [3]

$$C = J_+J_- + J_3^2 - J_3 + 2\beta J_3^2(J_3 - 1)^2 \tag{2}$$

have been constructed, leading to

$$\begin{aligned}J_+|j,m\rangle &= ((j-m)(j+m+1+2\gamma)(1+2\beta(j(j+1)+m(m+1)\\&\qquad +2\gamma(j+m+1+\gamma))))^{\frac{1}{2}}|j,m+1\rangle\\J_-|j,m\rangle &= J_+^+|j,m\rangle = ((j-m+1)(j+m+2\gamma)(1+2\beta(j(j+1)+m(m-1)\\&\qquad +2\gamma(j+m+\gamma))))^{\frac{1}{2}}|j,m-1\rangle\\J_3|j,m\rangle &= (m+\gamma)|j,m\rangle\\C|j,m\rangle &= ((j+\gamma)(j+\gamma+1)+2\beta(j+\gamma)^2(j+\gamma+1)^2)|j,m\rangle\end{aligned} \tag{3}$$

where $\{|j,m\rangle\}$ is the $(2j+1)$-dimensional basis of the ordinary sl(2)-algebra (corresponding to $\beta = 0$). In fact, three families of representations appeared, characterized respectively by

$$\gamma = 0 \tag{4}$$

$$\gamma = \pm\frac{1}{2\beta}(-\beta - 4\beta^2 j(j+1))^{\frac{1}{2}} \tag{5}$$

and the associated constraints on the parameter β

$$\beta \geqslant -\frac{1}{4j^2} \tag{6}$$

$$\frac{-1}{4j(j+1)} < \beta \leqslant \frac{-1}{4j(j+1)+1} \tag{7}$$

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for equations (4) and (5), respectively.

Let us notice that the representations corresponding to (3), (4) and (6) have been previously mentioned by Zhedanov [3] while the representations dealing with (3), (5) and (7) were new ones. This $\gamma = 0$ context is also the only one for which the Casimir operator of the Higgs algebra is expressed in terms of the sl(2) one only

$$C = C^{\mathrm{sl}(2)}(1 + 2\beta C^{\mathrm{sl}(2)}). \tag{8}$$

We propose here to generalize the results (3) through

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{f(m)}|j,m+c\rangle \\ J_-|j,m\rangle &= J_+^+|j,m\rangle = \sqrt{f(m-c)}|j,m-c\rangle \\ J_3|j,m\rangle &= \left(\frac{m}{c} + \gamma\right)|j,m\rangle \end{aligned} \tag{9}$$

where c is a nonnegative (and nonvanishing) integer. The two first commutation relations of (1) are automatically satisfied while the third one asks for the recurrence relation

$$f(m) = f(m-c) - 2\left(\frac{m}{c} + \gamma\right) - 8\beta\left(\frac{m}{c} + \gamma\right)^3. \tag{10}$$

The lowest weight constraint

$$f(-j-c+n) = 0 \qquad n = 0, 1, \ldots, c-1 \tag{11}$$

coupled with the relation (10), leads to

$$\begin{aligned} f(m) = -2\sum_{k=0}^{\frac{m+j-n}{c}}\left(-\frac{j}{c} + k + \gamma + \frac{n}{c}\right) - 8\beta\sum_{k=0}^{\frac{m+j-n}{c}}\left(-\frac{j}{c} + k + \gamma + \frac{n}{c}\right)^3 \\ m = -j+n, -j+n+c, \ldots \end{aligned} \tag{12}$$

or, in other words, to

$$\begin{aligned} f(m) = \frac{1}{c^2}(m+j-n+c)(j-m-n-2\gamma c)\Big(1 + \frac{\beta}{c^2}(2j^2 + 2m^2 + 2n^2 - 4jn + 2jc \\ +2mc - 2nc - 4j\gamma c + 4n\gamma c + 4m\gamma c + 4\gamma^2 c^2)\Big) \\ m = -j+n, -j+n+c, \ldots \qquad n = 0, 1, \ldots, c-1. \end{aligned} \tag{13}$$

As a result, the eigenvalues of the Casimir operator (2) now read

$$\begin{aligned} C|j,m\rangle = \frac{1}{c^2}(j-n-\gamma c)(j-n-\gamma c+c) \\ \times\left(1 + \frac{2\beta}{c^2}(j-n-\gamma c)(j-n-\gamma c+c)\right)|j,m\rangle. \end{aligned} \tag{14}$$

Let us note that, up to the interchanges

$$\gamma \leftrightarrow -\gamma \tag{15}$$

$$j - m \leftrightarrow j + m + 1 \tag{16}$$

the first one being justified by the 'symmetry' of γ (see (4) and (5)) and the second one being related to the lowest weight–highest weight connection for sl(2), we recover, for $c = 1$, the results (3). For $c = 2, 3, \ldots$, we have additional representations with respect to those obtained in [1]. In particular, supplementary values of β are now available, a fact which could be of interest in connection with the physical applications [2] subtended by the Higgs algebra (β being related to the curvature of the two-dimensional space we are working in).

Indeed, if we consider the following additional constraints on f

$$f(j) = f(j-1) = \cdots = f(j-c+1) = 0 \tag{17}$$

arising in order to keep $\{|j, m\rangle\}$ as the $(2j+1)$-dimensional basis of the usual angular momentum theory, we are led, for example, to

$$\gamma = 0 \qquad \beta = -c^2 \qquad \text{or} \qquad \gamma = \pm\frac{1}{2c} \qquad \beta = -\frac{c^2}{4} \tag{18}$$

if $j = \frac{1}{2}$, $c = 2, 3, \ldots$. Consequently, some more values $(-4, -9, -\frac{9}{4}, \ldots)$ appear with respect to the set implied by equations (5) and (6) and we evidently obtain additional unitary irreducible representations besides those constructed in [1].

Let us illustrate this result on a specific example. We consider $j = \frac{3}{2}$. Following [1], β is constrained as

$$\beta \geqslant -\tfrac{1}{9} \tag{19}$$

$$-\tfrac{1}{15} < \beta \leqslant -\tfrac{1}{16} \tag{20}$$

in the vanishing and nonvanishing γ contexts, respectively. In particular, the $\beta = -\frac{4}{13}$ value is excluded in both cases. Now if we take account of the generalized bosons b and $b^\dagger$ introduced by Brandt and Greenberg [4] satisfying

$$[N, b] = -2b \qquad [b, b^\dagger] = 1 \qquad M = b^\dagger b \tag{21}$$

and acting on the usual Fock states as follows

$$N|n\rangle = n|n\rangle \tag{22}$$

$$\begin{aligned} M|2m\rangle &= m|2m\rangle \\ M|2m+1\rangle &= m|2m+1\rangle \end{aligned} \tag{23}$$

we easily see that

$$J_+ = \sqrt{\frac{6}{13}}\, b_1 b_2^\dagger \qquad J_- = \sqrt{\frac{6}{13}}\, b_2 b_1^\dagger \qquad J_3 = \tfrac{1}{4}(N_2 - N_1) \tag{24}$$

(1 and 2 referring to two independent bosonic systems) is an *ad hoc* realization of the Higgs algebra corresponding to $\beta = -\frac{4}{13}$. In fact, it coincides with the representations (9), (13) where $c = 2$ and $\gamma = 0$.

As a final comment, we would like to mention that the set (9)–(17) is not the only possible generalization of (13). In particular, we could relax the constraint (17) and obtain new (lowest-weight) infinite-dimensional representations of the Higgs algebra. For instance, if we take $c = 1$ (and $f(j) \neq 0$), we are led to the relation (3) (with $m = -j, -j+1, \ldots$) and the constraints

$$\beta \leqslant \frac{-1}{4(j+1)^2} \qquad \text{if } \gamma = 0 \tag{25}$$

$$\beta \leqslant \frac{-1}{4} \qquad \text{if } \gamma = -j-1 \tag{26}$$

$$\beta \leqslant \frac{-1}{4(j+1+\gamma)^2} \qquad \text{if } \gamma > -j-1 \text{ (but } \neq 0). \tag{27}$$

In this case, the Higgs generators can be expressed in terms of the sp$(2, R)$ generators, K_3, $K_\pm$ such that

$$[K_3, K_\pm] = \pm K_\pm$$
$$[K_+, K_-] = -2K_3. \tag{28}$$

Indeed, in the $\gamma = 0$ context, for example, we have

$$J_3 = K_3 \tag{29}$$

and

$$J_+ = K + \sqrt{-1 - 2\beta(C^{\mathrm{sp}(2)} + K_3^2 + K_3)} \tag{30}$$

if $C^{\mathrm{sp}(2)}$ is the usual sp$(2, R)$ Casimir operator.

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References

[1] Abdesselam B, Beckers J, Chakrabarti A and Debergh N 1996 *J. Phys. A: Math. Gen.* **29** 3075
[2] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309
[3] Zhedanov A S 1992 *Mod. Phys. Lett.* A **7** 507
[4] Brandt R A and Greenberg O W 1969 *J. Math. Phys.* **10** 1168